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2009 J. Phys. A: Math. Theor. 42 265203

(http://iopscience.iop.org/1751-8121/42/26/265203)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 265203 (15pp) doi:10.1088/1751-8113/42/26/265203

Hierarchy of QM SUSYs on a bounded domain

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Received 26 December 2008, in final form 24 April 2009
Published 8 June 2009
Online at stacks.iop.org/JPhysA/42/265203

Abstract
We systematically formulate a hierarchy of isospectral Hamiltonians in one-dimensional supersymmetric quantum mechanics on an interval and on a circle, in which two successive Hamiltonians form $\mathscr{N} = 2$ supersymmetry. We find that boundary conditions compatible with supersymmetry are severely restricted. In the case of an interval, a hierarchy of, at most, three isospectral Hamiltonians is possible with unique boundary conditions, while in the case of a circle an infinite tower of isospectral Hamiltonians can be constructed with a two-parameter family of boundary conditions.

PACS numbers: 11.30.Pb, 03.65.−w, 03.65.Fd

1. Introduction

Although, historically, supersymmetric quantum mechanics (SUSY QM) was originally introduced by Witten [1] as a toy model for studying patterns of supersymmetry breakings, it was soon recognized that SUSY QM was interesting in its own right; for example, it provides a systematic description of categorizing analytically solvable potentials using the so-called shape invariance (see [2] for review). Schrödinger equations with shape-invariant potentials can be solved algebraically with the aid of supersymmetry. SUSY QM also appears in various contexts of physics; it is related to soliton physics [3–8] including inverse scattering problems [9–11], two-dimensional quantum field theories [12, 13], supersymmetric lattice models leaving time direction continuous [14], integrable models such as the Calogero model and its application to black hole physics [15–19], and quantum mechanics with point singularities [20, 21].

Recently it was shown that in higher dimensional gauge theories with extra compact dimensions, there always exists an $\mathscr{N} = 2$ quantum mechanical supersymmetry (QM SUSY) in the 4D spectrum; the Kaluza–Klein mass eigenvalue problems are equivalent to energy eigenvalue problems in $\mathcal{N} = 2$ SUSY QM [22]. The $\mathcal{N} = 2$ QM SUSY can be regarded as a remnant of the higher dimensional gauge invariance, and plays an essential role to generate an infinite tower of massive spin-1 particles. In [23], it was pointed out that a hierarchical mass spectrum can naturally arise in the context of a higher dimensional gauge theory with a warped metric and give a solution to the gauge hierarchy problem, in which the $\mathcal{N} = 2$ QM SUSY turns out to play a crucial role. Since the extra dimension is compactified, the corresponding supersymmetric quantum mechanical systems are of course constrained to bounded domains. There, boundary conditions are very important not only for the infrared regime but also for the ultraviolet regime, and play an essential role in determining the 4D particle spectrum especially for the low energy levels or massless mode. When the compactified dimension does not respect the translational invariance due to the presence of extended defects (branes or boundaries), boundary effects also play a significant role in the ultraviolet regime as boundary-localized divergent terms [24]. Such localized ultraviolet divergences must be renormalized by field theory operators on the boundary and give rise to nontrivial renormalization group flows for brane-localized theory [25, 26]. Since any gauge-invariant field theory possesses the $\mathcal{N} = 2$ QM SUSY, the boundary conditions and the $\mathcal{N} = 2$ QM SUSY must be compatible with each other. In this paper, we will address this issue from the supersymmetric quantum mechanics' point of view: we analyze the possible boundary conditions in one-dimensional $\mathcal{N} = 2$ SUSY QM on a bounded domain $(0, L)$.

The analysis developed in [22] was extended to 5D gravity [27]. In 5D gravity, it was shown that *two* $\mathcal{N} = 2$ SUSYs are hidden in the 4D spectrum. The two $\mathcal{N} = 2$ SUSYs can be regarded as a remnant of higher dimensional general coordinate invariance, and are needed in order for the 'Higgs' mechanism to generate massive spin-2 particles; one of the two quantum mechanical SUSYs ensures the degeneracy between spin-2 and spin-1 excitations and the other between spin-1 and spin-0 excitations. A crucial ingredient of this coexistence of two quantum mechanical SUSYs is the refactorization of Hamiltonians (Laplace operators). In view of these facts, it would be natural to guess that in a higher dimensional spin-N field theory there would exist N $\mathcal{N} = 2$ SUSYs in the 4D mass spectrum. In this paper, we will also investigate whether it is possible to construct such a hierarchy of N SUSYs without conflicting with the boundary conditions.

The rest of this paper is organized as follows. In section 2, we analyze the possible boundary conditions in $\mathcal{N} = 2$ SUSY QM on a bounded domain $(0, L)$. We show that the allowed boundary conditions in $\mathcal{N} = 2$ SUSY QM are limited to the so-called scale-independent subfamily of the $U(2)$ family of boundary conditions [28]. In section 3, we construct a hierarchy of N SUSYs by solving the refactorization condition. The results coincide with the so-called isospectral deformations of the Hamiltonian [29–31]. In section 4, we analyze the allowed boundary conditions of the quantum mechanical system with N SUSYs on an interval and on a circle separately and present a systematic prescription to construct a hierarchy of isospectral Hamiltonians. Section 5 is devoted to conclusions and discussions.

2. Boundary conditions in $\mathcal{N}$ = 2 SUSY QM

Hermiticity of Hamiltonians is the basic principle in quantum theory; it leads to the unitarity of the S-matrix or the conservation of probability in the whole quantum system. In one-dimensional non-supersymmetric quantum mechanics, it is known that the most general boundary conditions consistent with the hermiticity of Hamiltonians are characterized by a 2×2 unitary matrix U [28]. In one-dimensional $\mathcal{N} = 2$ SUSY QM, however, supersymmetry imposes more severe constraints on the parameter space of this $U(2)$ family of boundary conditions. As we will show below, the possible boundary conditions consistent with $\mathcal{N} = 2$ supersymmetry are limited to the so-called scale-independent subfamily of the $U(2)$ family of boundary conditions.

To begin with let us consider $\mathcal{N} = 2$ SUSY QM on a finite domain $(0, L) \in \mathbb{R}$, whose Hamiltonians are given by[4]

$$H_0 = Q_0^\dagger Q_0, \tag{1a}$$

$$H_1 = Q_0 Q_0^\dagger. \tag{1b}$$

The supercharge Q_0 and its adjoint $Q_0^\dagger$ are given by

$$Q_0 = \frac{\mathrm{d}}{\mathrm{d}x} + W_0'(x), \tag{2a}$$

$$Q_0^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + W_0'(x), \tag{2b}$$

where W_0 is a superpotential (or prepotential), which must be a real function in order to guarantee the hermiticity of Hamiltonians, and the prime ($'$) indicates the derivative with respect to x. In terms of the zero-mode function $\phi_0^{(0)}$ satisfying the equation $Q_0\phi_0^{(0)} = 0$, the superpotential W_0 can be written as

$$W_0(x) = -\ln \phi_0^{(0)}(x). \tag{3}$$

Supersymmetric relations are

$$Q_0\phi_0 = \sqrt{E}\phi_1, \tag{4a}$$

$$Q_0^\dagger\phi_1 = \sqrt{E}\phi_0, \tag{4b}$$

where ϕ_0 and ϕ_1 are eigenfunctions of H_0 and H_1, respectively, with the common energy E. In this paper, we will concentrate on a finite superpotential on the whole domain. In other words, we require that $\phi_0^{(0)}$ has no zero point (or no node).

Next, we will focus on the hermiticity of H_0 and then derive the allowed boundary conditions for ϕ_0 and ϕ_1 using the supersymmetric relations (4a) and (4b) respectively. In physical language, the hermiticity of Hamiltonian H_0 indicates the conservation of probability in the whole system $j_0(0) = j_0(L)$, where the probability current density j_0 is defined by $j_0 = -\mathrm{i}\big(\big(\phi_0^*\big)'\phi_0 - \phi_0^*\phi_0'\big)$. It is more suitable for the following discussion to rewrite the probability current density in the following form:

$$j_0(x) = -\mathrm{i}\big[(Q_0\phi_0)^*(x)\phi_0(x) - \phi_0^*(x)(Q_0\phi_0)(x)\big], \tag{5}$$

which follows from the real-valued superpotential.

There are two physically distinct cases as follows.

[4] $\mathcal{N} = 2$ supersymmetry will be transparent by introducing the following 2×2 matrix operators:

$$\mathscr{H} = \begin{bmatrix} H_0 & 0 \\ 0 & H_1 \end{bmatrix}, \qquad (-1)^F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathscr{Q}_1 = \begin{bmatrix} 0 & Q_0^\dagger \\ Q_0 & 0 \end{bmatrix}, \qquad \mathscr{Q}_2 = \mathrm{i}(-1)^F\mathscr{Q}_1,$$

which satisfy the standard $\mathcal{N} = 2$ supersymmetry algebra

$$\{\mathscr{Q}_i, \mathscr{Q}_j\} = 2\delta_{ij}\mathscr{H}, \qquad [\mathscr{Q}_i, \mathscr{H}] = 0, \qquad [(-1)^F, \mathscr{H}] = 0, \qquad \{(-1)^F, \mathscr{Q}_i\} = 0, \qquad i, j = 1, 2.$$

(1) *Case* $j_0(0) = 0 = j_0(L)$.
In this case, the probability current density j_0 does not flow outside the domain and the probability is locally conserved. Hence, the two ends of the domain $x = 0$ and L are physically disconnected and we will refer to this case as an interval case.

(2) *Case* $j_0(0) = j_0(L) (\neq 0)$.
In this case j_0 flows outside the domain but the probability is globally conserved as an entire system, which implies that the two ends of the domain are physically connected. Hence, we will refer to this case as a circle case. Although in this case the end points $x = 0$ and L are physically identified, there is no need for the superpotential W_0 to be a periodic function; when the superpotential does not have a periodicity of L, there just arises some kind of singularity at the junction point $x = 0$, which can be characterized by the boundary conditions just as in the point interactions [28].

In the following subsections, we will study these two cases separately.

2.1. Interval case: $j_0(0) = 0 = j_0(L)$

We first investigate the condition $j_0(0) = 0 = j_0(L)$. Note that the condition $j_0(x_i) = 0$ $(i = 1, 2; x_1 = 0, x_2 = L)$ can be written as follows:

$$|\phi_0(x_i) - \mathrm{i}L_0(Q_0\phi_0)(x_i)|^2 = |\phi_0(x_i) + \mathrm{i}L_0(Q_0\phi_0)(x_i)|^2, \tag{6}$$

where L_0 is an arbitrary real constant of mass dimension -1, which is just introduced to adjust the mass dimension of the equation. As we will see below, L_0 is not a parameter characterizing the boundary conditions.

The above equation implies that the two complex numbers $\phi_0(x_i) - \mathrm{i}L_0(Q_0\phi_0)(x_i)$ and $\phi_0(x_i) + \mathrm{i}L_0(Q_0\phi_0)(x_i)$ are different from each other at most only in a phase factor. Thus, we can write

$$\phi_0(x_i) - \mathrm{i}L_0(Q_0\phi_0)(x_i) = \mathrm{e}^{\mathrm{i}\theta_i}(\phi_0(x_i) + \mathrm{i}L_0(Q_0\phi_0)(x_i)), \tag{7}$$

where $0 \leqslant \theta_i < 2\pi, i = 1, 2$. When one considers a non-supersymmetric quantum mechanics, this becomes the end of the story by just replacing the supercharge Q_0 with the ordinary derivative $\mathrm{d}/\mathrm{d}x$, and the resulting boundary conditions are parameterized by the group $U(1) \times U(1)$, whose parameter space is a 2-torus $S^1 \times S^1 \simeq T^2$ [28]. However, supersymmetry severely restricts the allowed parameter space. Using the supersymmetric relations (4*a*) and (4*b*), we find

$$\sin\left(\frac{\theta_i}{2}\right)\phi_0(x_i) + L_0\cos\left(\frac{\theta_i}{2}\right)(Q_0\phi_0)(x_i) = 0, \tag{8a}$$

$$\sin\left(\frac{\theta_i}{2}\right)\left(Q_0^\dagger\phi_1\right)(x_i) + EL_0\cos\left(\frac{\theta_i}{2}\right)\phi_1(x_i) = 0. \tag{8b}$$

Since the boundary conditions should not depend on the eigenvalue E (otherwise the superposition of the quantum states becomes meaningless), the parameters θ_i $(i = 1, 2)$ must be 0 or π. Thus in $\mathcal{N} = 2$ SUSY QM on an interval the boundary conditions compatible with the supersymmetry are characterized by the discrete group $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset U(1) \times U(1)$, which just consists of four 0-dimensional points $\{\mathrm{e}^{\mathrm{i}0}, \mathrm{e}^{\mathrm{i}\pi}\} \times \{\mathrm{e}^{\mathrm{i}0}, \mathrm{e}^{\mathrm{i}\pi}\} = \{1, -1\} \times \{1, -1\}$. This result is consistent with the previous analyses of SUSY QM with point singularities [20, 21]. Now it is clear that the allowed boundary conditions can be categorized into the following $2 \times 2 = 4$ types:

$$(\theta_1, \theta_2) = (0, 0) : \begin{cases} (Q_0\phi_0)(0) = 0 = (Q_0\phi_0)(L), \\ \phi_1(0) = 0 = \phi_1(L); \end{cases} \tag{9a}$$

$$(\theta_1, \theta_2) = (\pi, \pi) : \begin{cases} \phi_0(0) = 0 = \phi_0(L), \\ \left(Q_0^\dagger\phi_1\right)(0) = 0 = \left(Q_0^\dagger\phi_1\right)(L); \end{cases} \tag{9b}$$

$$(\theta_1, \theta_2) = (0, \pi) : \begin{cases} (Q_0\phi_0)(0) = 0 = \phi_0(L), \\ \phi_1(0) = 0 = \left(Q_0^\dagger\phi_1\right)(L); \end{cases} \tag{9c}$$

$$(\theta_1, \theta_2) = (\pi, 0) : \begin{cases} \phi_0(0) = 0 = (Q_0\phi_0)(L), \\ \left(Q_0^\dagger\phi_1\right)(0) = 0 = \phi_1(L). \end{cases} \tag{9d}$$

2.2. *Circle case:* $j_0(0) = j_0(L) (\neq 0)$

SUSY QM on a circle or with periodic potentials has been vastly studied in the literature. Most of the previous works concern the construction of new (quasi-)exactly solvable models [32–46] or non-Hermitian $\mathcal{PT}$-symmetric quantum mechanics as a SUSY partner system [47–52]. No systematic description has been, however, made on possible boundary conditions consistent with the hermiticity of each Hamiltonian and the supersymmetry. In this subsection, we clarify the most general boundary conditions compatible with the requirement for the probability conservation $j_0(0) = j_0(L) (\neq 0)$ as well as the SUSY relations (4). The condition $j_0(0) = j_0(L)$ can be written in the following form:

$$\left|\Phi_{\phi_0} - \mathrm{i}L_0\sigma_3\Phi_{Q_0\phi_0}\right|^2 = \left|\Phi_{\phi_0} + \mathrm{i}L_0\sigma_3\Phi_{Q_0\phi_0}\right|^2, \tag{10}$$

where for any function $f(x)$ the two-component boundary value vector Φ_f is defined as

$$\Phi_f := \begin{bmatrix} f(0) \\ f(L) \end{bmatrix}. \tag{11}$$

σ_3 is the third Pauli matrix: $\sigma_3 = \mathrm{diag}(1, -1)$. This equation shows that the squared length of the two-dimensional complex column vector $\Phi_{\phi_0} - \mathrm{i}L_0\sigma_3\Phi_{Q_0\phi_0}$ is equal to that of $\Phi_{\phi_0} + \mathrm{i}L_0\sigma_3\Phi_{Q_0\phi_0}$, which implies that these two vectors must be related by a two-dimensional unitary transformation. Thus, we can write

$$\Phi_{\phi_0} - \mathrm{i}L_0\sigma_3\Phi_{Q_0\phi_0} = U\left(\Phi_{\phi_0} + \mathrm{i}L_0\sigma_3\Phi_{Q_0\phi_0}\right), \tag{12}$$

where U is an arbitrary 2×2 unitary matrix. In one-dimensional non-supersymmetric quantum mechanics, it is known that the most general boundary conditions are characterized by this $U(2)$ family [28]. In the following, we shall determine the possible form of this unitary matrix compatible with supersymmetry and find the allowed subspace of the $U(2)$ family.

To this end, we first apply the supersymmetric relations to condition (12). Using the supersymmetric relations (4*a*) and (4*b*), we find

$$(\mathbb{1} - U)\Phi_{\phi_0} - \mathrm{i}L_0(\mathbb{1} + U)\sigma_3\Phi_{Q_0\phi_0} = \vec{0}, \tag{13a}$$

$$(\mathbb{1} - U)\Phi_{Q_0^\dagger\phi_1} - \mathrm{i}EL_0(\mathbb{1} + U)\sigma_3\Phi_{\phi_1} = \vec{0}. \tag{13b}$$

Again since the boundary conditions should not depend on the eigenvalue E, the eigenvalues of the matrix U must be 1 or -1, which is equivalent to the condition $U^2 = \mathbb{1}$. Note that any unitary matrix satisfying $U^2 = \mathbb{1}$ can be spectrally decomposed using the projection operators $P_+ = \frac{1}{2}(\mathbb{1} + U)$ and $P_- = \frac{1}{2}(\mathbb{1} - U)$, which satisfy $P_+ + P_- = \mathbb{1}$, $(P_\pm)^2 = \mathbb{1}$ and $P_\pm P_\mp = 0$. Multiplying these projection operators, the above boundary conditions boil down to the following four independent conditions:

$$(\mathbb{1} - U)\Phi_{\phi_0} = \vec{0}, \tag{14a}$$

$$(\mathbb{1} + U)\sigma_3\Phi_{Q_0\phi_0} = \vec{0}, \tag{14b}$$

$$(\mathbb{1} - U)\Phi_{Q_0^\dagger\phi_1} = \vec{0}, \tag{14c}$$

$$(\mathbb{1} + U)\sigma_3\Phi_{\phi_1} = \vec{0}. \tag{14d}$$

Note that when $U = \mathbb{1}$ ($U = -\mathbb{1}$), these boundary conditions reduce to type $(0, 0)$ (type (π, π)) boundary conditions in the interval case and lead to $j_0(0) = 0 = j_0(L)$. Thus in this circle case these two 'points' $U = \mathbb{1}$ and $-\mathbb{1}$ have to be removed from the parameter space, from which we conclude that the two eigenvalues of U must be 1 and -1 respectively. Such a unitary matrix can be written as follows:

$$U = \vec{e}\cdot\vec{\sigma}, \tag{15}$$

where $\vec{\sigma}$ are the Pauli matrices and $\vec{e}$ is a unit vector, which can be parameterized as

$$\vec{e} = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi), \qquad 0 \leqslant \theta < 2\pi, \qquad 0 \leqslant \phi \leqslant \pi. \tag{16}$$

Note that when $\phi = 0$ ($\phi = \pi$), that is, $U = \sigma_3$ ($U = -\sigma_3$), the boundary conditions become type $(0, \pi)$ (type $(\pi, 0)$) boundary conditions in the interval case and again lead to $j_0(0) = 0 = j_0(L)$. Thus in the circle case these two 'points' $U = \sigma_3$ and $-\sigma_3$, which correspond to the north pole $\phi = 0$ and the south pole $\phi = \pi$ of S^2, respectively, must be removed from the parameter space S^2. The resulting parameter space is thus isomorphic to a non-compact two-dimensional cylinder. In summary the boundary conditions compatible with $\mathscr{N} = 2$ supersymmetry have a two-parameter family, which can be written as

$$\begin{bmatrix} \phi_0(L) \\ (Q_0\phi_0)(L) \end{bmatrix} = \mathrm{e}^{\mathrm{i}\theta}\begin{bmatrix} \tan(\phi/2) & 0 \\ 0 & \cot(\phi/2) \end{bmatrix}\begin{bmatrix} \phi_0(0) \\ (Q_0\phi_0)(0) \end{bmatrix}, \tag{17a}$$

$$\begin{bmatrix} \phi_1(L) \\ \big(Q_0^\dagger\phi_1\big)(L) \end{bmatrix} = \mathrm{e}^{\mathrm{i}\theta}\begin{bmatrix} \cot(\phi/2) & 0 \\ 0 & \tan(\phi/2) \end{bmatrix}\begin{bmatrix} \phi_1(0) \\ \big(Q_0^\dagger\phi_1\big)(0) \end{bmatrix}, \tag{17b}$$

where $0 \leqslant \theta < 2\pi$ and $0 < \phi < \pi$. In practical calculations, it is convenient to introduce a real parameter η defined as

$$\mathrm{e}^\eta := \tan\left(\frac{\phi}{2}\right), \qquad -\infty < \eta < \infty. \tag{18}$$

Before closing this section, we should make a comment on the physical meanings of these two parameters θ and η. As is well known, θ corresponds to the magnetic flux penetrating through the circle (see for example [53]). On the other hand, as shown in [54], boundary conditions with nonzero η corresponds to the presence of δ'-singularity at the junction point $x = 0$.

3. Refactorization of Hamiltonians

As already mentioned in section 1, quantum mechanical supersymmetry plays an essential role in generating massive Kaluza–Klein particles in higher dimensional field theory. It has been shown that in 5D gravity, *two* $\mathscr{N} = 2$ quantum mechanical SUSYs are needed in order for the 'Higgs' mechanism to generate massive spin-2 particles [27]. A crucial ingredient of this coexistence of two quantum mechanical SUSYs is the refactorization of Hamiltonians. Thus, it would be natural to guess that in a higher dimensional spin-N field theory there would exist a hierarchy of N SUSYs in the 4D mass spectrum, whose typical structure must be

$$\begin{aligned}
H_0 &= Q_0^\dagger Q_0 \\
H_1 &= Q_0 Q_0^\dagger = Q_1^\dagger Q_1 + c_1 \\
H_2 &\qquad\quad\; = Q_1 Q_1^\dagger + c_1 = Q_2^\dagger Q_2 + c_1 + c_2 \\
H_3 &\qquad\qquad\qquad\qquad\quad\; = Q_2 Q_2^\dagger + c_1 + c_2 \\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

where the nth supercharge and its adjoint are assumed to be of the form

$$Q_n = \mathrm{e}^{-W_n(x)} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{+W_n(x)} = \frac{\mathrm{d}}{\mathrm{d}x} + W_n'(x), \tag{19a}$$

$$Q_n^\dagger = -\mathrm{e}^{+W_n(x)} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{-W_n(x)} = -\frac{\mathrm{d}}{\mathrm{d}x} + W_n'(x), \tag{19b}$$

and c_n is a real constant. In the context of higher dimensional field theory, W_n and c_n would correspond to the warp factor and the cosmological constant on 3-branes, respectively.

In this section, we solve the refactorization condition of Hamiltonians in the case of $c_n = 0$ and construct a hierarchy of supersymmetry.

3.1. Refactorization of Hamiltonians

Although in this paper we will focus on the case that all the constant shifts c_n are zero, it may be instructive to keep c_n to be nonzero in order to distinguish our refactorization method and the conventional one, which is used to solve the Schrödinger equation by the method of shape invariance.

The refactorization condition for the nth Hamiltonian $Q_{n-1} Q_{n-1}^\dagger = Q_n^\dagger Q_n + c_n$ can be written in the following form:

$$(W_{n-1}')^2 + W_{n-1}'' = (W_n')^2 - W_n'' + c_n. \tag{20}$$

This is a recursion relation known as the ladder equation in the context of parasupersymmetric or higher derivative supersymmetric quantum mechanics [55–61]. Our task is to solve equation (20) with respect to W_n and to recursively define the nth superpotential. The nonlinear differential equation (20) is the Riccati equation in terms of W_n so that it can be linearized as follows:

$$Q_{n-1} Q_{n-1}^\dagger \mathrm{e}^{-W_n} = c_n \mathrm{e}^{-W_n} \tag{21}$$

or, equivalently,

$$H_n \mathrm{e}^{-W_n} = \left(\sum_{i=1}^{n} c_i \right) \mathrm{e}^{-W_n}. \tag{22}$$

This is nothing but the Schrödinger equation for the nth Hamiltonian. Noting that the spectrum of the nth Hamiltonian is bounded from below by the constant $\sum_{i=1}^{n} c_i$, we see that equation (22) is the Schrödinger equation for the ground state.

When $c_n = 0$, it is easy to solve equation (21) with the result

$$W_n = -W_{n-1} - \ln \left\{ \alpha_{n-1} + \beta_{n-1} \int_{x_0}^{x} \mathrm{d}y \, \mathrm{e}^{-2W_{n-1}(y)} \right\}, \tag{23}$$

where α_n and β_n are integration constants. x_0 is an arbitrary point placed on the interval $(0, L)$. Since in this paper we concentrate on finite superpotentials even at the boundaries, it is convenient to choose x_0 as $x_0 = 0$ and β_{n-1} as $\beta_{n-1} = \left[\int_0^L \mathrm{d}y \exp(-2W_{n-1})\right]^{-1}$. We note that a constant shift of the superpotentials has no effect on the Hamiltonians. With these choices, the parameter α_{n-1} is limited to the ranges $\alpha_{n-1} < -1$ and $0 < \alpha_{n-1}$ for the well-definedness of W_n'. Thus, once given a quantum mechanical system, we can always construct an infinite hierarchy of Hamiltonians.

Note that result (23) coincides with the so-called isospectral deformations of the Hamiltonian [29–31].

3.2. Three-term recurrence relation for nonzero modes

Let $\phi_n^{(l)}$ be the energy eigenfunction of lth excited states for the nth Hamiltonian. Then, we have the three-term recurrence relation for quantum mechanical systems with N SUSYs:

$$\phi_{n+2}^{(l)} = -\phi_n^{(l)} + \frac{1}{\sqrt{E_l}}(W_n' + W_{n+1}')\phi_{n+1}^{(l)}, \tag{24}$$

which follows from the SUSY relations $\sqrt{E_l}\phi_{n+2}^{(l)} = Q_{n+1}\phi_{n+1}^{(l)}$, $\sqrt{E_l}\phi_n^{(l)} = Q_n^\dagger\phi_{n+1}^{(l)}$ and the identity $Q_{n+1} = -Q_n^\dagger + W_n' + W_{n+1}'$. Note that when $\beta_{n+1} = 0$, $\phi_{n+2}^{(l)}$ just reduces to the (opposite sign of) energy eigenfunction $\phi_n^{(l)}$.

3.3. Zero mode

Next, we will show that the zero-mode functions $\phi_n^{(0)}$ for $0 < n < N$ cannot exist in general in a quantum mechanical system with N SUSYs. To this end, suppose that we have constructed a set of $N + 1$ isospectral Hamiltonians using the refactorization method. Since the nth Hamiltonian H_n can be written in two ways as $H_n = Q_{n-1}Q_{n-1}^\dagger = Q_n^\dagger Q_n$, $\phi_n^{(0)}(x)$ with $n = 1, \ldots, N-1$ has to satisfy the equations

$$Q_{n-1}^\dagger\phi_n^{(0)} = 0 = Q_n\phi_n^{(0)} \tag{25}$$

or, equivalently,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - W_{n-1}'\right)\phi_n^{(0)} = 0 = \left(\frac{\mathrm{d}}{\mathrm{d}x} - W_{n-1}' - \frac{\beta_{n-1}\,\mathrm{e}^{-2W_{n-1}}}{\alpha_{n-1} + \beta_{n-1}\int_{x_0}^x \mathrm{d}y\,\mathrm{e}^{-2W_{n-1}}}\right)\phi_n^{(0)}. \tag{26}$$

Obviously, there is no nontrivial solution to these two different equations except for the case $\beta_{n-1} = 0$. When $\beta_{n-1} = 0$, the $(n+1)$ th Hamiltonian $H_{n+1} = Q_{n+1}^\dagger Q_{n+1}$ comes to be identical to the $(n-1)$ th Hamiltonian, which has no interest for us. Therefore, there is no nontrivial solution to (25). We thus conclude that the zero-mode solutions consistent with N SUSYs can exist *at most* only for the case $n = 0$ and N. The ground-state energy eigenfunction for H_N is obtained by solving the equation $Q_{N-1}^\dagger\phi_N^{(0)} = 0$, which can be easily integrated with the result

$$\phi_N^{(0)}(x) = C\mathrm{e}^{+W_{N-1}(x)}, \tag{27}$$

where C is the normalization constant. If $\phi_N^{(0)}$ turns out not to obey the boundary conditions, only a single zero mode $\phi_0^{(0)}$ exists. A typical spectrum of a quantum mechanical system with N SUSYs is shown in figure 1.

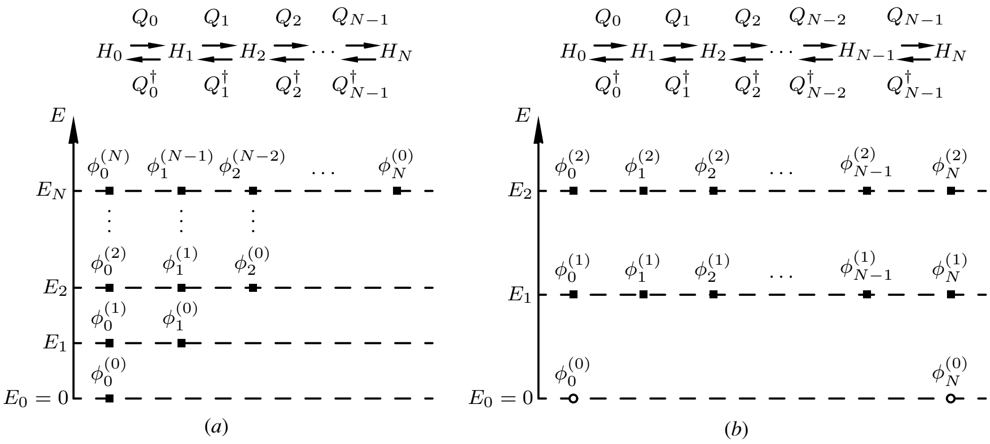

Figure 1. (*a*) Typical spectrum of a quantum system constructed by the conventional refactorization method with $W_n = -\ln \phi_n^{(0)}$. (*b*) Typical spectrum of a quantum system with N SUSYs.

4. Hierarchy of QM SUSYs

In the previous section, we have not discussed boundary conditions compatible with N SUSYs. In this section, we will investigate whether it is possible to construct a hierarchical SUSY without conflicting with the hermiticity of each Hamiltonian. In the subsequent subsections, we will study this hierarchical SUSY on an interval and on a circle separately.

4.1. Hierarchy on an interval

Let us first study a hierarchical SUSY on an interval. As a first step, let us consider the boundary conditions consistent with two SUSYs. Inserting the supersymmetric relations $Q_1\phi_1 = \sqrt{E}\phi_2$ and $Q_1^\dagger\phi_2 = \sqrt{E}\phi_1$ into equation (8*a*), we have

$$\phi_0 : 0 = \sin\left(\frac{\theta_i}{2}\right)\phi_0(x_i) + L_0\cos\left(\frac{\theta_i}{2}\right)(Q_0\phi_0)(x_i), \tag{28a}$$

$$\phi_1 : 0 = \sin\left(\frac{\theta_i}{2}\right)\big(Q_0^\dagger\phi_1\big)(x_i) + EL_0\cos\left(\frac{\theta_i}{2}\right)\phi_1(x_i), \tag{28b}$$

$$\begin{aligned}\phi_2 : 0 = {}& \sin\left(\frac{\theta_i}{2}\right)(W_0' + W_1')(x_i)\big(Q_1^\dagger\phi_2\big)(x_i)\\ &+ E\left\{-\sin\left(\frac{\theta_i}{2}\right)\phi_2(x_i) + L_0\cos\left(\frac{\theta_i}{2}\right)\big(Q_1^\dagger\phi_2\big)(x_i)\right\},\end{aligned} \tag{28c}$$

where the third equation follows from equation (28*b*) with the identity $Q_0^\dagger = -Q_1 + W_0' + W_1'$. Now it is obvious that there are no possible boundary conditions independent of E except for the choice $\theta_i = 0$. Thus, the boundary conditions consistent with two SUSYs are uniquely determined as follows:

$$(Q_0\phi_0)(x_i) = 0, \tag{29a}$$

$$\phi_1(x_i) = 0, \tag{29b}$$

$$\left(Q_1^\dagger\phi_2\right)(x_i) = 0. \tag{29c}$$

It is easy to show that there are no possible boundary conditions consistent with a hierarchy of N SUSYs for $N \geqslant 3$. Thus, we conclude that, at most, three successive quantum mechanical systems on an interval can be supersymmetric in a hierarchy of QM SUSYs.

4.2. Hierarchy on a circle

Let us next study a hierarchical SUSY on a circle. As mentioned before in this paper, we focus on finite superpotentials on the whole domain. When W_0 is finite, the finite $(n+1)$ th superpotential W_{n+1} is recursively defined as

$$W_{n+1}(x) = -W_n(x) - \ln\left[\alpha_n + \beta_n \int_0^x \mathrm{d}y\, \mathrm{e}^{-2W_n(y)}\right], \qquad \text{for} \qquad n = 0, 1, 2, \ldots, \tag{30}$$

with

$$\alpha_n < -1 \qquad \text{or} \qquad 0 < \alpha_n, \qquad \beta_n = \left[\int_0^L \mathrm{d}x\, \mathrm{e}^{-2W_n(x)}\right]^{-1}. \tag{31}$$

Since the hierarchy of N SUSYs is just the assembly of $\mathscr{N} = 2$ SUSYs, the boundary conditions in the H_n–H_{n+1} sector have to be of the form

$$\begin{bmatrix} \phi_n(L) \\ (Q_n\phi_n)(L) \end{bmatrix} = \mathrm{e}^{\mathrm{i}\theta_n} \begin{bmatrix} \mathrm{e}^{\eta_n} & 0 \\ 0 & \mathrm{e}^{-\eta_n} \end{bmatrix} \begin{bmatrix} \phi_n(0) \\ (Q_n\phi_n)(0) \end{bmatrix}, \tag{32a}$$

$$\begin{bmatrix} \phi_{n+1}(L) \\ \left(Q_n^\dagger\phi_{n+1}\right)(L) \end{bmatrix} = \mathrm{e}^{\mathrm{i}\theta_n} \begin{bmatrix} \mathrm{e}^{-\eta_n} & 0 \\ 0 & \mathrm{e}^{\eta_n} \end{bmatrix} \begin{bmatrix} \phi_{n+1}(0) \\ \left(Q_n^\dagger\phi_{n+1}\right)(0) \end{bmatrix}, \tag{32b}$$

with

$$0 \leqslant \theta_n < 2\pi \qquad \text{and} \qquad -\infty < \eta_n < \infty. \tag{33}$$

For the sake of concreteness of the discussion, let us first consider two SUSYs in the H_0–H_1–H_2 sector. The point is whether there exists a well-defined parameter region to be consistent with two different boundary conditions for the wavefunction $\phi_1(x)$ of the middle Hamiltonian system H_1:

$$\phi_1(L) = \mathrm{e}^{\mathrm{i}\theta_0 - \eta_0}\phi_1(0), \tag{34a}$$

$$\left(Q_0^\dagger\phi_1\right)(L) = \mathrm{e}^{\mathrm{i}\theta_0 + \eta_0}\left(Q_0^\dagger\phi_1\right)(0), \tag{34b}$$

which come from equation (32b) for $n = 0$, and

$$\phi_1(L) = \mathrm{e}^{\mathrm{i}\theta_1 + \eta_1}\phi_1(0), \tag{35a}$$

$$(Q_1\phi_1)(L) = \mathrm{e}^{\mathrm{i}\theta_1 - \eta_1}(Q_1\phi_1)(0), \tag{35b}$$

which come from equation (32a) for $n = 1$.

First, it is obvious that the parameters θ_1 and η_1 have to be equal to θ_0 and $-\eta_0$, respectively:

$$\theta_1 = \theta_0, \qquad \eta_1 = -\eta_0. \tag{36}$$

Next, by adding equations (34a) and (35b)

$$(W_0'(L) + W_1'(L))\phi_1(L) = \mathrm{e}^{\mathrm{i}\theta_0 + \eta_0}(W_0'(0) + W_1'(0))\phi_1(0), \tag{37}$$

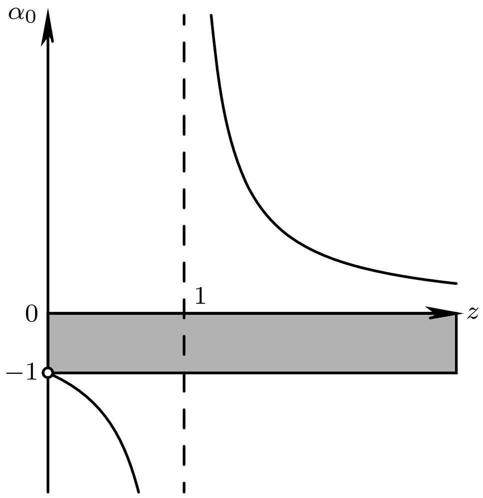

Figure 2. Allowed region of the isospectral parameter α_0 as a function of $z = \exp[-2(\eta_0 + \int_0^L \mathrm{d}x\, W_0'(x))]$, whose range is $0 < z < \infty$.

from which we find

$$\mathrm{e}^{2\eta_0} = \frac{W_0'(L) + W_1'(L)}{W_0'(0) + W_1'(0)}$$
$$= \frac{\alpha_0}{1+\alpha_0} \exp\left(-2\int_0^L \mathrm{d}x\, W_0'(x)\right), \tag{38}$$

where the last equality follows from equation (30). Thus in order to implement the two boundary conditions, the isospectral parameter α_0 has to be tuned as

$$\alpha_0{}^{-1} = \exp\left[-2\left(\eta_0 + \int_0^L \mathrm{d}x\, W_0'(x)\right)\right] - 1. \tag{39}$$

Note that once the parameters η_1 and α_0 are tuned as equations (36) and (39) respectively, the following identity holds:

$$\eta_1 + \int_0^L \mathrm{d}x\, W_1'(x) = \eta_0 + \int_0^L \mathrm{d}x\, W_0'(x). \tag{40}$$

The above procedure can be easily continued to arbitrary n. The resulting boundary conditions are as follows:

$$\phi_n(L) = \mathrm{e}^{\mathrm{i}\theta_0 \pm \eta_0} \phi_n(0), \tag{41a}$$
$$(Q_n\phi_n)(L) = \mathrm{e}^{\mathrm{i}\theta_0 \mp \eta_0} (Q_n\phi_n)(0), \tag{41b}$$

where the + (−) sign is for $n = 0, 2, 4\ldots$ ($n = 1, 3, 5\ldots$). The isospectral parameters are tuned as

$$\alpha_n{}^{-1} = \exp\left[-2\left(\eta_0 + \int_0^L \mathrm{d}x\, W_0'(x)\right)\right] - 1, \qquad n = 0, 1, 2, \ldots, \tag{42}$$

where α_n takes a desired value of $\alpha_n < -1$ or $\alpha_n > 0$ (see figure 2), as it should be. We thus conclude that starting from any quantum mechanical system on a circle, we can systematically construct an infinite hierarchy of QM SUSYs. We should emphasize the difference between the hierarchy on an interval and that on a circle. In the hierarchy on an interval, at most, three successive quantum mechanical systems can be supersymmetric with the unique boundary conditions (29*a*)–(29*c*). On the other hand, in the hierarchy on a circle, we can obtain an infinite tower of quantum mechanical systems whose successive two systems form an $\mathscr{N} = 2$ SUSY with the boundary conditions (41*a*) and (41*b*), which are specified by two parameters θ_0 and η_0 respectively.

5. Conclusions and discussions

In this paper, we have clarified the possible boundary conditions in $\mathscr{N} = 2$ supersymmetric quantum mechanics on a finite domain $(0, L)$ without conflicting with the conservation of probability current. The allowed boundary conditions in $\mathscr{N} = 2$ supersymmetric quantum mechanics are limited to the so-called scale-independent subfamily of the $U(2)$ family of boundary conditions. We also studied the hierarchy of N SUSYs and showed that in an interval case, it is not possible to construct beyond two SUSYs. On the other hand, in a circle case it is possible to construct an infinite hierarchy of supersymmetries by tuning the isospectral parameters α_n.

Let us close with some remarks.

(i) *Loop effects of* η. We show that in $\mathscr{N} = 2$ supersymmetric quantum mechanics on a circle, it is possible to introduce two parameters θ and η into the boundary conditions. As mentioned in section 2, θ corresponds to the magnetic flux penetrating through the circle and nonzero η corresponds to the presence of the δ'-singularity at the junction point $x = 0$. In higher dimensional gauge theory compactified on a circle, it is widely known that the twisted boundary conditions give rise to gauge symmetry/supersymmetry breaking known as the Hosotani/Scherk–Schwarz mechanism. However, the effect of the presence of η is not yet fully understood. It is interesting to investigate the loop effects of the parameter η in five-dimensional gauge theory with a single extra dimension compactified on a circle. We will address this issue elsewhere.

(ii) *Integrable models*. As opposed to the shape-invariant method, the techniques developed in this paper cannot be used to solve the Schrödinger equation. However, once given a solvable model, it is possible to generate an infinite tower of isospectral solvable models with nontrivial potential energy terms.

(iii) *Spin-N field theory*. In this paper, we formulate a systematic description for constructing the hierarchy of N SUSYs and show that in an interval case it is not possible to construct beyond two SUSYs. Since it seems a necessary condition in order to generate massive Kaluza–Klein particles, one might expect that it is possible to prove some kind of no-go theorem of the 'Higgs' mechanism for spin-N ($\geqslant 3$) particle in the context of five-dimensional field theory with a single extra dimension compactified on an interval. However, this is an open question.

(iv) *Relax to* $\mathcal{PT}$*-symmetry*. Recently, a considerable number of studies have been made on non-Hermitian $\mathcal{PT}$-symmetric quantum mechanics (see [62] for recent review). It is known that the conventional hermiticity condition on the Hamiltonian is a sufficient condition for the real and lower bounded spectra and can be replaced by the weaker condition of the $\mathcal{PT}$-symmetry of the Hamiltonian. In this paper, we impose the hermiticity of Hamiltonian; however, it is interesting to relax the hermiticity condition to the $\mathcal{PT}$-symmetric one. But it is not clear to the authors how to treat the $\mathcal{PT}$-symmetry into the boundary conditions.

(v) *Exceptional cases*. In this paper, we require that the zero-mode function has no zero point (or no node). This is equivalent to the statement that our analysis is limited to the non-singular potential which does not diverge even at the boundaries. Once relaxing this limitation, we know by experience that it is possible to construct a hierarchy of SUSY Hamiltonians beyond two successive steps without conflicting with the hermiticity of each Hamiltonian even in an interval system. This exception comes from the fact that wavefunctions can simultaneously satisfy *two distinct* boundary conditions, i.e. the Dirichlet and Neumann ones at the boundaries where the potential diverges. (This does not happen for non-singular potentials.) Then, the arguments in subsection 4.1 cannot be applied to these cases. It would be of great interest to extend our analysis for singular potentials.

Acknowledgments

One of the authors (SO) would like to thank S Odake for very useful discussions and his hospitality. This work is supported in part by the Grant-in-Aid for Scientific Research (no. 18540275) by the Japanese Ministry of Education, Science, Sports and Culture.

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